2010
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time - 5 minutes.
- Working Time - 3 hours.
- Write using a blue or black pen.
- Board Approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question in a new booklet

Total marks (**120**)

- Attempt Questions 1-8.
- All questions are of equal value.

2010 Extension 2 Trial Higher School Certificate

Question 1 (15 Marks) **Marks**

a) Find $\int \frac{x}{\sqrt{16-9x^2}}\, dx$ **1**

b) Find $\int_0^{\pi} x \sin x \; dx$ **3**

c) Use the substitution $u = \sqrt{x-1}$ to evaluate $\int_2^3 \frac{1+x}{\sqrt{x-1}}\, dx$ **3**

d) Use partial fractions to find $\int \frac{(2x^2+5x+9)dx}{(x-1)(x^2+2x+5)}$ **4**

e) Use the substitution $t = \tan\frac{\theta}{2}$ to show that $\int_0^{\frac{\pi}{3}} \frac{1}{1+\sin\theta}\, d\theta = \sqrt{3} - 1$ **4**

Question 2 (15 Marks) Begin a New Booklet

a) Given $A = 3 + 4i$ and $B = 1 - i$, express the following in the form $x + iy$

(i) AB **2**

(ii) $\dfrac{A}{iB}$ **2**

(iii) $\sqrt{A}$ **3**

b) Let $\alpha = \sqrt{3} - i$

(i) Find the exact value of $|\alpha|$ and $\arg \alpha$ **2**

(ii) Find the exact value of α^5 in the form $a + ib$ where a and b are real. **2**

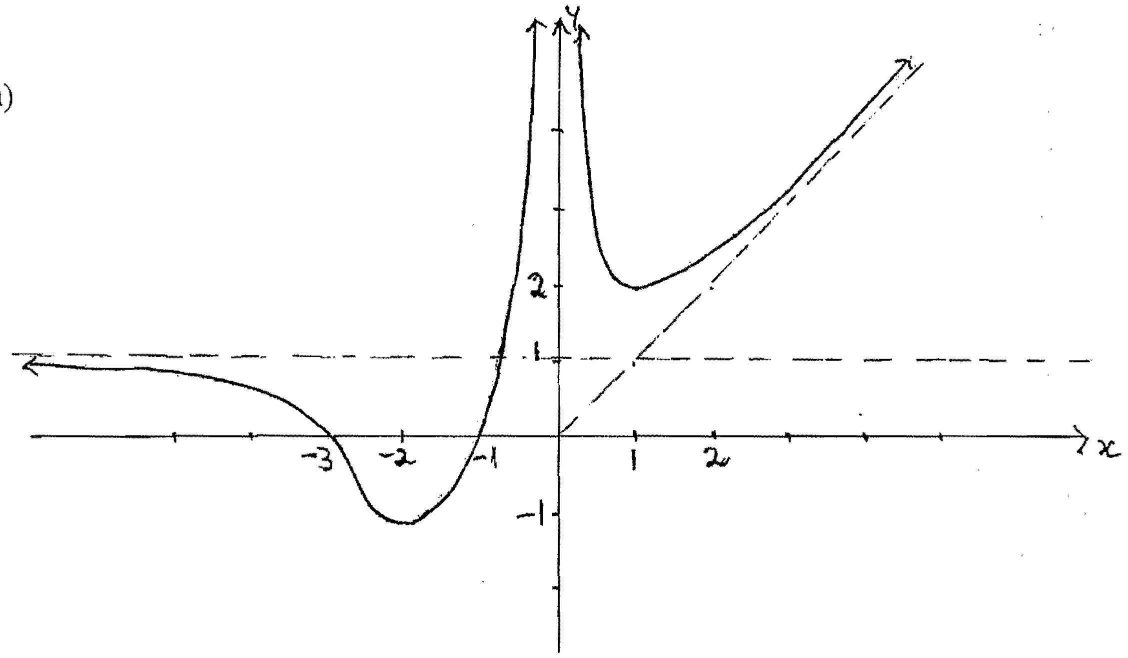
c) (i) On an Argand diagram, shade the region where

$$|z - 1 - i| \le \sqrt{2} \quad \text{and} \quad 0 \le \arg z \le \frac{\pi}{4} \quad \text{both hold.}$$ **2**

(ii) Find, in simplest exact form, the area of this shaded region. **2**

Question 3 (15 Marks) Begin a New Booklet **Marks**

a)



The diagram shows the graph of $y = f(x)$.

Draw separate half page sketches of:

(i) $y = (f(x))^2$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y^2 = f(x)$ **1**

(iv) $y = \dfrac{1}{f(x)}$ **2**

(v) $y = f'(x)$ **2**

b) For the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

(i) find the eccentricity **1**

(ii) find the coordinates of the foci S and S' **1**

(iii) find the equations of the directrices. **1**

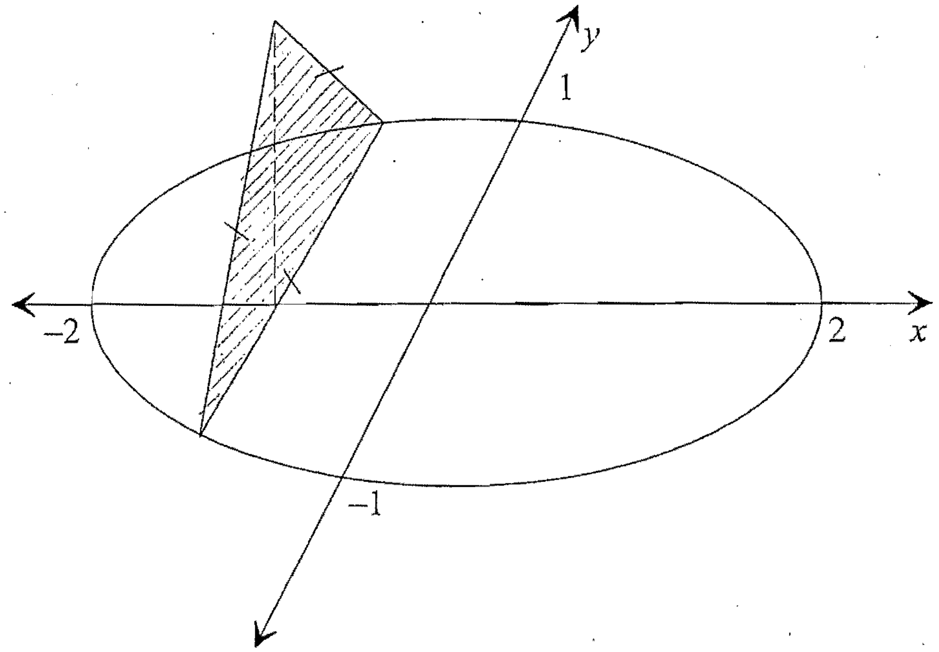
(iv) Sketch the curve showing the foci and directrices. **1**

P is an arbitrary point on this ellipse.

(v) Prove that the sum of the distances SP and $S'P$ is independent of P. **2**

Question 4 (15 Marks) Begin a New Booklet Marks

a) A solid shape has as its base an ellipse in the XY plane as shown below. Sections taken perpendicular to the X-axis are equilateral triangles. The major and minor axes of the ellipse are 4 metres and 2 metres respectively.



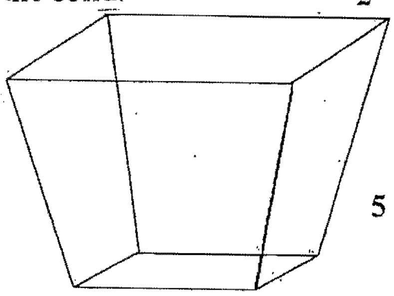
i. Write down the equation of the ellipse. 1

ii. Show that the area of the cross-section at $x = k$ is given by 2

$$A = \frac{\sqrt{3}}{4}\left(4 - k^2\right).$$

iii. By using the technique of slicing, find the volume of the solid. 2

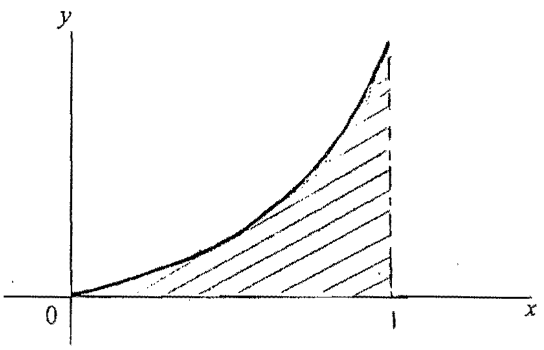
b) A container has 6 plane faces. The top is a rectangle, 70cm by 50cm. The bottom is a rectangle, 30cm by 20cm, parallel to the top rectangle. The remaining faces are all trapezia. The perpendicular height is 20cm.



Use slices parallel to the base to find the capacity of the container in litres 5

c) The area between the curve $y = e^{x^2} - 1$ and the x-axis, from $x = 0$ to $x = 1$ is rotated about the y-axis. 5

Sketch a typical cylindrical shell and use this method to find the volume formed.



Question 5 (15 Marks) Begin a New Booklet

a) The roots of the equation $x^3 - 3x^2 + 9 = 0$ are α, b and γ.

(i) Find the polynomial equation with roots α^2, β^2 and γ^2. **2**

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ and hence evaluate $\alpha^3 + \beta^3 + \gamma^3$ **2**

b) Given that the polynomial $P(x)$ has a double zero at $x = \alpha$, show that the polynomial $P'(x)$ will have a single zero at $x = \alpha$. **2**

c) When a polynomial $P(x)$ is divided by $(x-1)$ the remainder is 3 and when divided by $(x-2)$ the remainder is 5. Find the remainder when the polynomial is divided by $(x-1)(x-2)$. **3**

d) $P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$ and the equation $P(x) = 0$ has roots α, β, γ and δ.

(i) Show that the equation $P(x) = 0$ has no integer roots. **1**

(ii) Show that $P(x) = 0$ has a real root between 0 and 1. **1**

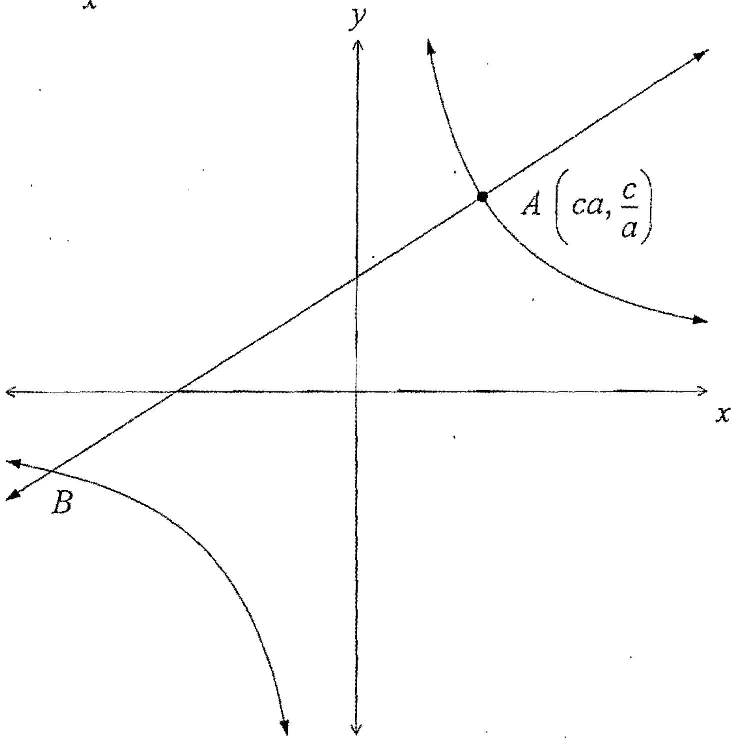
(iii) Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2$. **2**

(iv) Hence find the number of real roots of the equation $P(x) = 0$, giving reasons. **2**

Question 6 (15 Marks) Begin a New Booklet

a) Show that $\dfrac{x^4+x^2+1}{x^2} \geq 3$ for all x. (Hint: Consider $(x^2-1)^2$) **2**

b)



The point $A\left(ca, \dfrac{c}{a}\right)$, where $a \neq \pm 1$ lies on the hyperbola $xy = c^2$. The normal through A meets the other branch of the curve at B.

i. Show that the equation of the normal through A is **2**

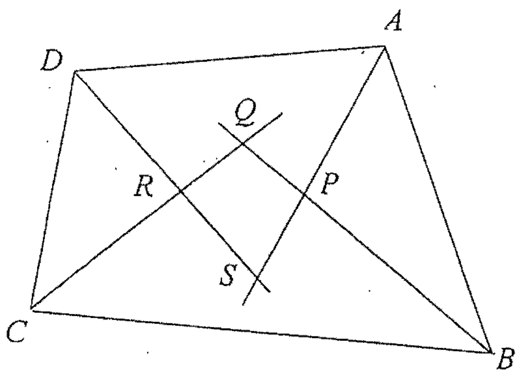
$$y = a^2x + \frac{c}{a}\left(1 - a^4\right)$$

ii. Hence if B has coordinates $\left(cb, \dfrac{c}{b}\right)$, show that $b = \dfrac{-1}{a^3}$. **3**

iii. If this hyperbola is rotated clockwise through 45°, show that the equation becomes **4**

$$x^2 - y^2 = 2c^2.$$

c)



In the quadrilateral $ABCD$ shown above, APS, BPQ, CRQ and DRS are the bisectors of the vertex angles at A, B, C and D respectively.

(i) Show that $PQRS$ is a cyclic quadrilateral. **2**

(ii) If $ABCD$ is a trapezium, deduce that one of the diagonals of $PQRS$ is a diameter of the circle through P, Q, R and S. **2**

Question 7 (15 Marks) Begin a New Booklet

a) A solid of unit mass is dropped under gravity from rest at a height of h metres. Air resistance is proportional to the speed v of the mass, acceleration under gravity is g.

(i) Using k as the constant of proportionality, show that the acceleration is given by **1**

$$\frac{d^2x}{dt^2} = g - kv$$

(ii) Show that the velocity v of the solid after t seconds is given by **3**

$$v = \frac{g}{k}(1 - e^{-kt})$$

(iii) Using the fact that $\frac{d^2x}{dt^2} = v\frac{dv}{dx}$ show that

$$x = \frac{g}{k^2}\left[\ln\frac{g}{g-kv} - \frac{kv}{g}\right]$$ **4**

b) Further from the surface of the Earth, we may neglect air resistance, but acceleration due to gravity is not constant. It varies inversely with the square of the distance from the centre of the Earth. That is

$$\frac{d^2x}{dt^2} = -\frac{k}{x^2}$$ where x is the distance measured from the centre of the Earth.

(i) If the acceleration due to gravity at the surface of the Earth is g, and the radius of the Earth is R, show that this constant is gR^2 **1**

(ii) A rocket is fired vertically upwards from the surface of the Earth with an initial velocity of u. Neglecting air resistance, show that its velocity v is given by

$$v^2 = \frac{2gR^2}{x} + u^2 - 2gR$$ **2**

(iii) Find (in terms of u, g, and R) the greatest height possible if the rocket is to return to Earth. **2**

(iv) Hence find the initial velocity which must be exceeded for the rocket to escape the Earth's gravitational pull and never return. **2**
(Use $g = 9.8 m/s^2$ and $R = 6367$ ***km***)

Question 8 (15 Marks) Begin a New Booklet

a) (i) Show that the recurrence (reduction) formula for

$$I_n = \int \tan^n x \; dx \quad \text{is} \quad I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$$ 4

(ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \tan^3 x \; dx$ 3

b) It is given that $z^5 = 1$ where $z \neq 1$

(i) Show that $z^2 + z + 1 + z^{-1} + z^{-2} = 0$ 1

(ii) Show that $z + z^{-1} = 2\cos\frac{2k\pi}{5}$ for $k = 1, 2, 3, 4$ 2

(iii) By letting $x = z + z^{-1}$ reduce the equation in (i) above to a quadratic equation in x. 3

(iv) Hence deduce that $\cos\frac{\pi}{5}.\cos\frac{2\pi}{5} = \frac{1}{4}$ 2

2010 Extension 2 Trial HSC

Q1

a) $\int \frac{x}{\sqrt{16-9x^2}}\,dx = \frac{-1}{18}\int -18x^1 (16-9x^2)^{-\frac{1}{2}}\,dx$

$= -\frac{1}{18}(16-9x^2)^{\frac{1}{2}}.\ 2$

$= -\frac{1}{9}\sqrt{16-9x^2} + C$

b) $\int_0^{\pi} x\sin x\,dx$ $\quad u = x \quad dv = \sin x\,dx$

$du = dx \quad v = -\cos x$

$\int u\,dv = uv - \int v\,du$

$\int_0^{\pi} x\sin x\,dx = \left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} -\cos x\,dx$

$= \left[-x\cos x + \sin x\right]_0^{\pi}$

$= -\pi.-1 + 0 + 0 - 0$

$= \pi$

c) $\int_2^3 \frac{1+x}{\sqrt{x-1}}\,dx$ $\quad u = \sqrt{x-1}$

or $u^2 = x-1$

$2u\,du = dx$

When $x=2 \quad u=1$

$x=3 \quad u=\sqrt{2}$

$= \int_1^{\sqrt{2}} \frac{(u^2+2).\ 2u\,du}{u}$

$= \int_1^{\sqrt{2}} 2u^2 + 4\,du$

$= \left[\frac{2u^3}{3} + 4u\right]_1^{\sqrt{2}}$

$= \frac{2}{3}.2\sqrt{2} + 4\sqrt{2} - \frac{2}{3} - 4$

$= \frac{16\sqrt{2}}{3} - \frac{14}{3}$ $\quad$ or $\frac{16\sqrt{2}-14}{3}$

d) Let $\frac{2x^2+5x+9}{(x-1)(x^2+2x+5)} \equiv \frac{a}{x-1} + \frac{bx+c}{x^2+2x+5}$

$$\equiv \frac{a(x^2+2x+5)+(bx+c)(x-1)}{(x-1)(x^2+2x+5)}$$

$\therefore 2x^2+5x+9 \equiv a(x^2+2x+5)+(bx+c)(x-1)$

Sub $x=1$: $16 = 8a \Rightarrow a=2$

Equate coeffs of x^2: $2 = a+b \Rightarrow b=0$

Sub $x=0$: $9 = 5a - c$

$9 = 10 - c \Rightarrow c=1$

$$\therefore \int \frac{2}{x-1} + \frac{1}{x^2+2x+5}\, dx$$

$$= \int \frac{2}{x-1} + \frac{1}{(x+1)^2+4}\, dx$$

$$= 2\ln|x-1| + \frac{1}{2}\tan^{-1}\frac{x+1}{2} + c$$

e) $\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\sin\theta}$ $t = \tan\frac{\theta}{2}$ $\theta=0$ $t=0$

$d\theta = \frac{2\,dt}{1+t^2}$ $\theta=\frac{\pi}{3}$

$t=\tan\frac{\pi}{6}$

$=\frac{1}{\sqrt{3}}$

$$\int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}} = \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{1+t^2+2t}$$

$$= \int 2(1+t)^{-2}\, dt$$

$$= \frac{2(1+t)^{-1}}{-1} = \left[\frac{-2}{1+t}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{-2}{1+\frac{1}{\sqrt{3}}} + 2 = 2 - \frac{2\sqrt{3}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = 2 - 3 + \sqrt{3}$$

$= 2$ $= \sqrt{3}-1$

Question 2

a) i) $(3+4i)(1-i) = 3-3i+4i+4$

$= \underline{7+i}$

ii) $\dfrac{3+4i}{i(1+i)} = \dfrac{3+4i}{i+1} \times \dfrac{1-i}{1-i}$

$= \dfrac{3-3i+4i+4}{1+1} = \dfrac{7+i}{2}$

$= \underline{\underline{\dfrac{7}{2} + \dfrac{i}{2}}}$

iii) Let $\sqrt{3+4i} = x+iy$

$3+4i = (x+iy)^2 = x^2-y^2+2xyi$

$\therefore \quad x^2-y^2 = 3$ —— ①

$2xy = 4$

$xy = 2$ —— ② $\qquad y = \dfrac{2}{x}$

$x^2 - \dfrac{4}{x^2} = 3$

$x^4 - 3x^2 - 4 = 0$

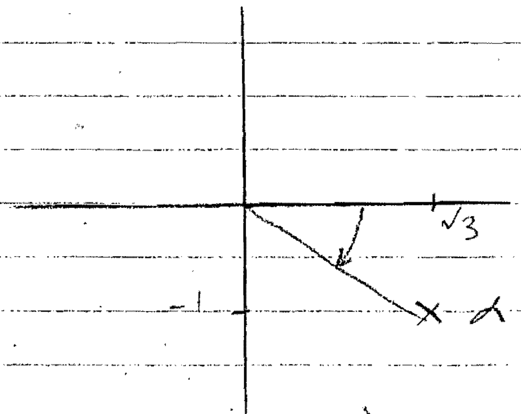
$(x^2-4)(x^2+1) = 0$

$x = \pm 2$ or $\pm i$ but x is real

$\therefore x = \pm 2 \quad y = \pm 1$

$\sqrt{A} = \underline{\underline{2+i \quad \text{or} \quad -2-i}}$

b) $\alpha = \sqrt{3} - i$



i) $|\alpha| = \sqrt{3+1} = 2$

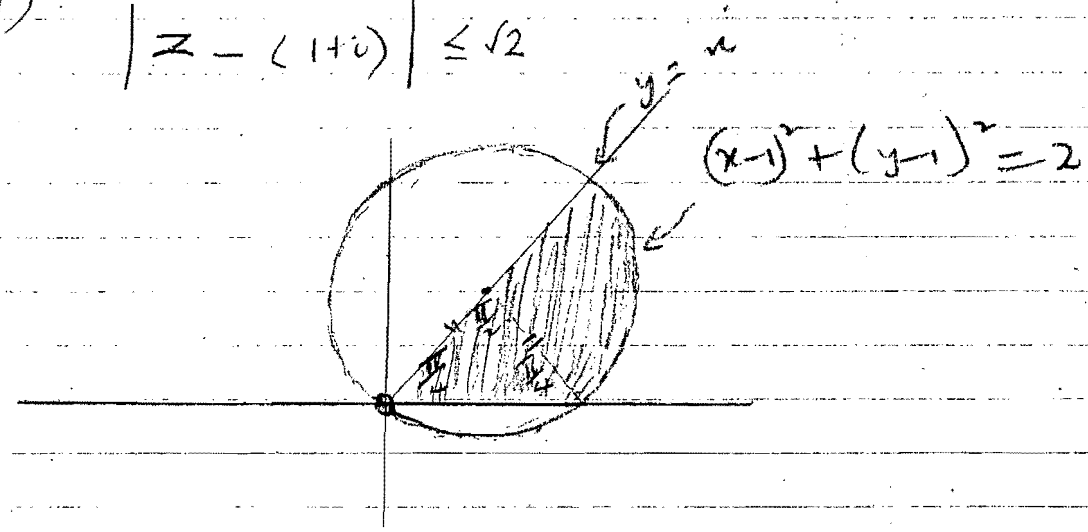
$\arg \alpha = -\dfrac{\pi}{6}$

ii) $\alpha^5 = \left\{2 \text{ cis } \dfrac{-\pi}{6}\right\}^5$

$= 32 \text{ cis } -\dfrac{5\pi}{6} = 32\left\{\cos \dfrac{-5\pi}{6} + i \sin \dfrac{-5\pi}{6}\right\}$

$= -32 \cos\dfrac{\pi}{6} - i \sin \dfrac{\pi}{6} = \underline{\underline{-16\sqrt{3} - 16i}}$

c) i) $|z-(1+i)| \le \sqrt{2}$



Area Minor Segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}(\sqrt{2})^2\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)$

$= \frac{\pi}{2} - 1$

Area Semi circle $= \frac{1}{2}\pi r^2$

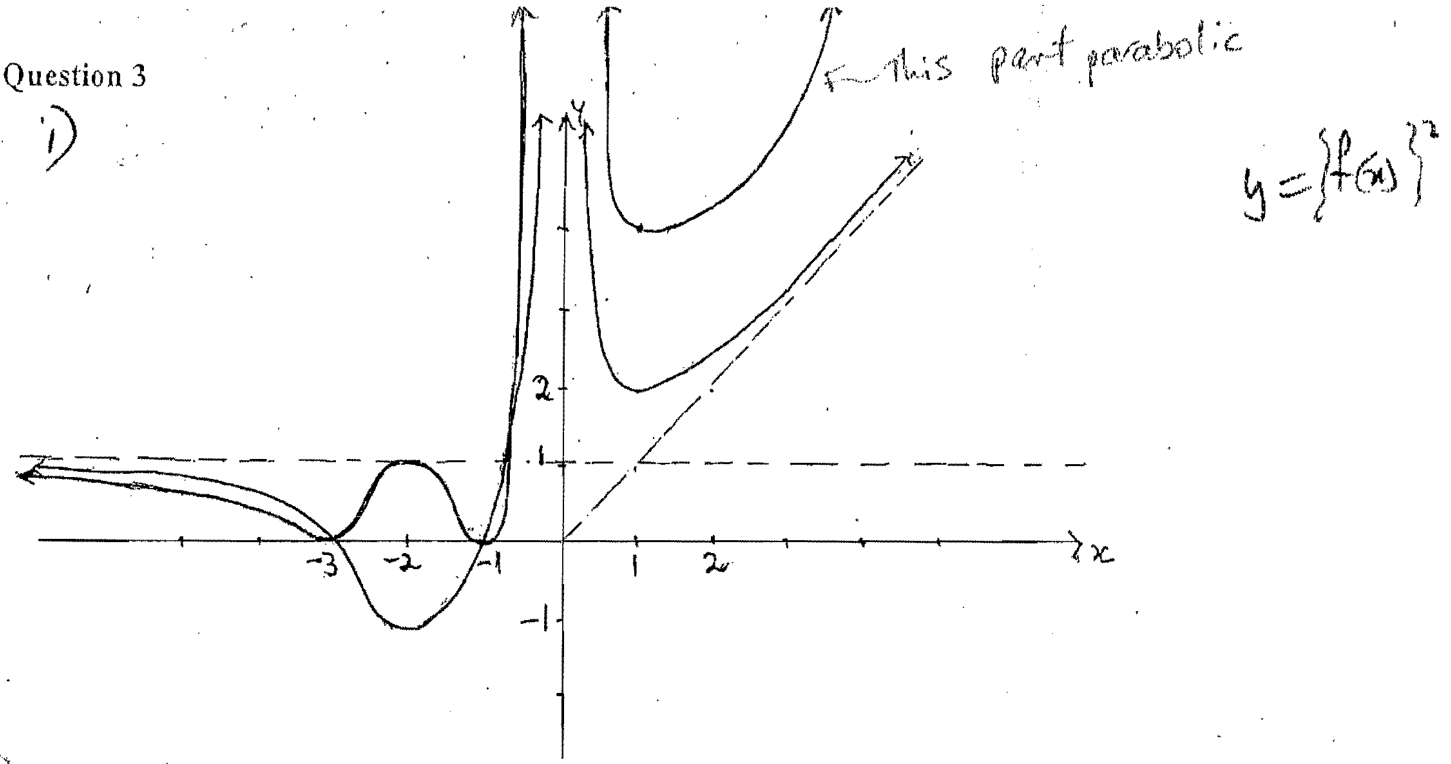
$= \frac{1}{2}\cdot\pi\cdot 2 \quad = \pi$

$\therefore$ Shaded area $= \pi - \left(\frac{\pi}{2} - 1\right)$

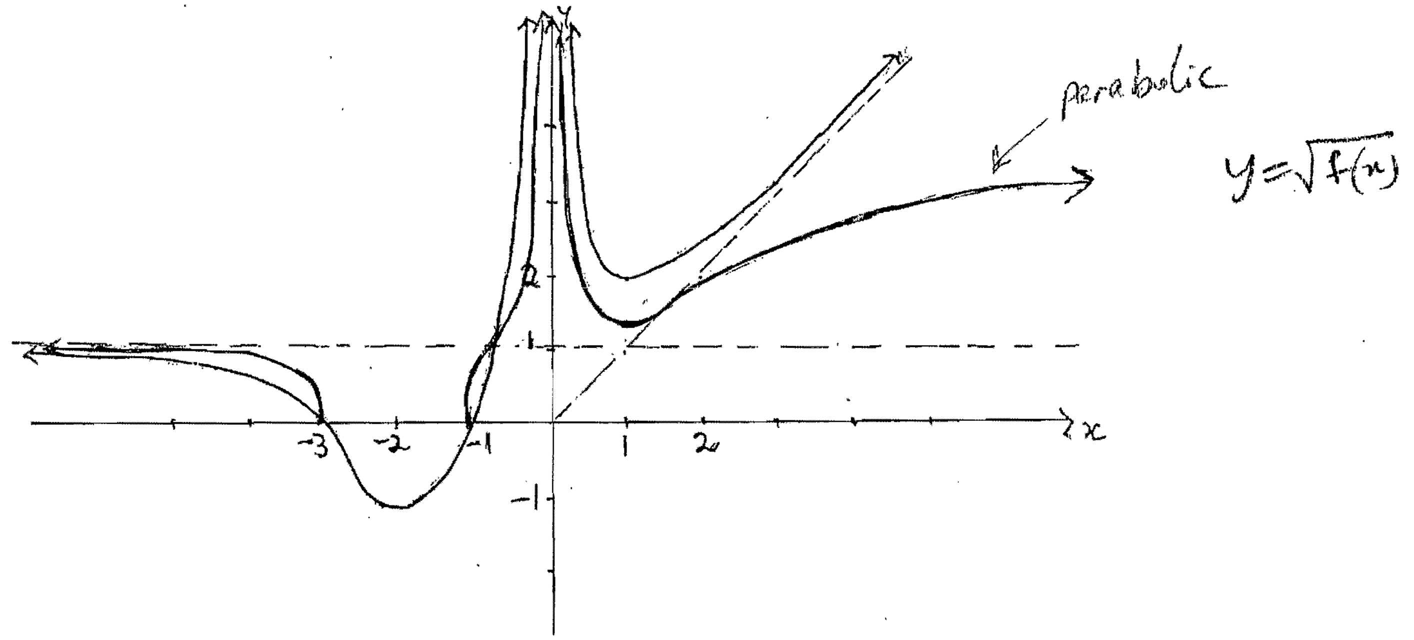
$= \frac{\pi}{2} + 1 \quad u^2$

Question 3

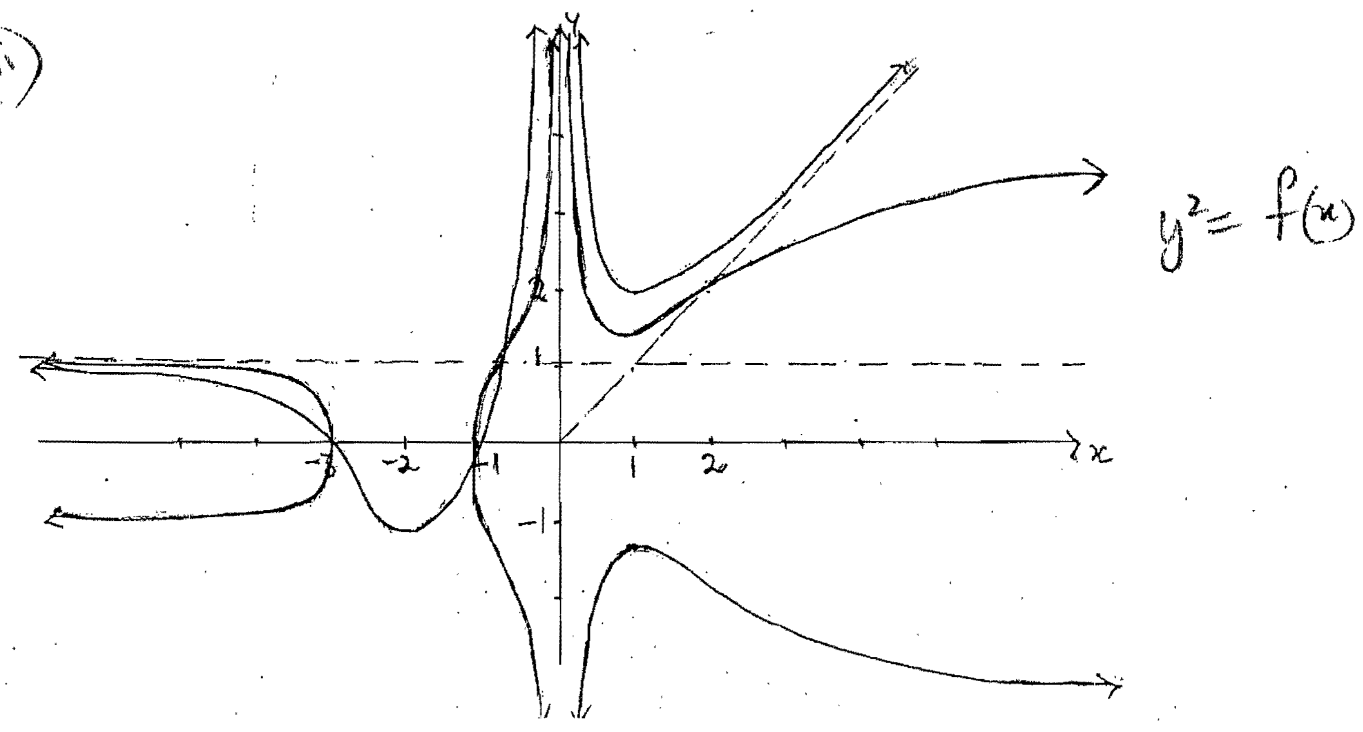
i)



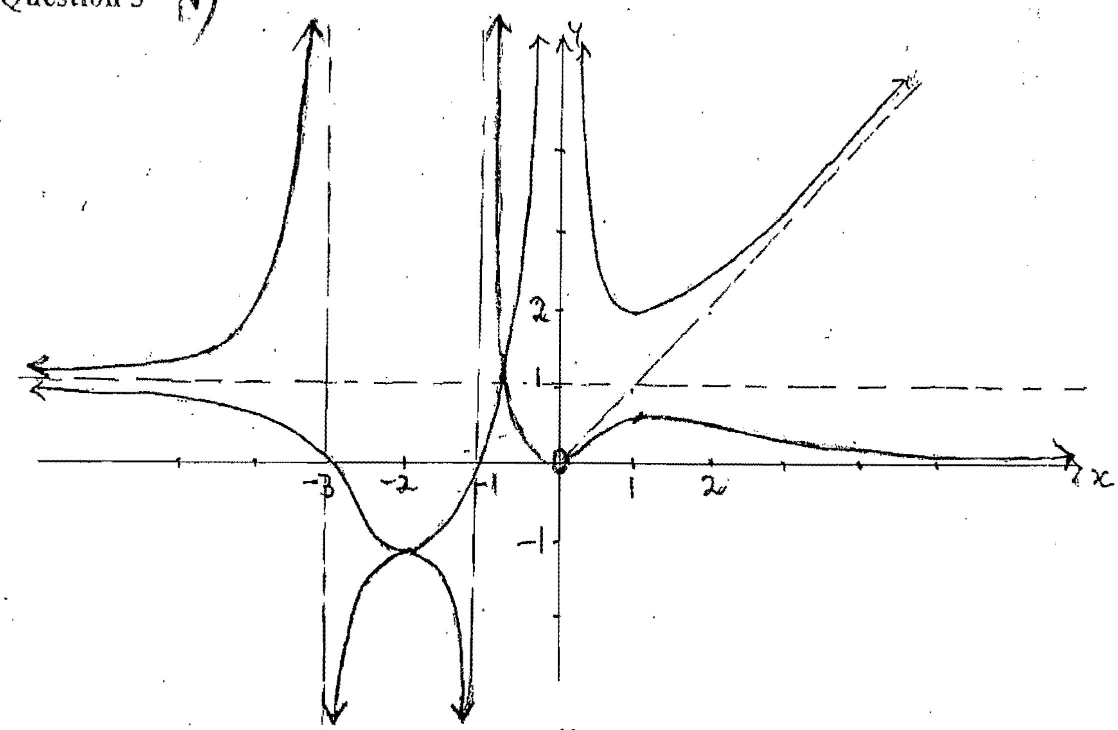
ii)



iii)

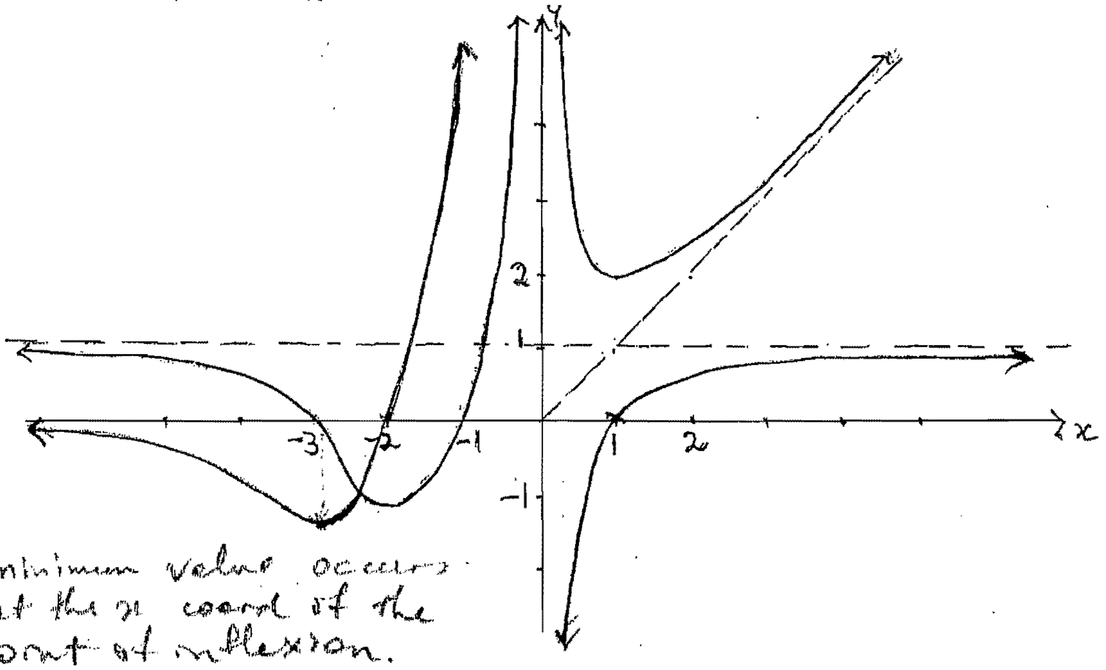


Question 3 iv)



$y = \frac{1}{f(x)}$

v)



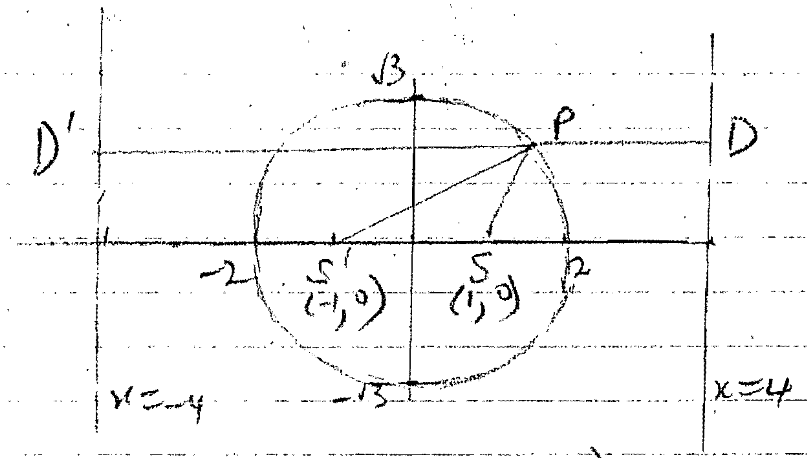
$y = f'(x)$

minimum value occurs at the x coord of the point of inflexion.

3

b) $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$a = 2, \quad b = \sqrt{3}$



(iv)

i) $b^2 = a^2(1-e^2)$

$3 = 4(1-e^2)$

$\frac{3}{4} = 1 - e^2$

$e^2 = \frac{1}{4}$

$e = \frac{1}{2}$ (e is positive)

ii) S $(ae, 0)$ S' $(-ae, 0)$

S $(1, 0)$ S' $(-1, 0)$

iii) $x = \pm \frac{a}{e} \implies x = \pm 4$

v) $PS + PS' = ePD + ePD'$

from definition of the ellipse as the locus of points whose distance from the focus is e times distance from the directrix

$= e(PD + PD')$

$SP + S'P = 8e$

ie independent of P.

Question 4

a) i) $\frac{x^2}{4} + \frac{y^2}{1} = 1$

ii) At $x = k$, $y^2 = 1 - \frac{k^2}{4}$

$y = \pm\sqrt{\frac{4-k^2}{4}} = \pm\frac{1}{2}\sqrt{4-k^2}$

$\therefore$ length of side $= 2y = \sqrt{4-k^2}$

Area $= \frac{1}{2}ab\sin C = \frac{1}{2}\sqrt{4-k^2}\sqrt{4-k^2}\sin 60°$

$= \frac{1}{2}(4-k^2) \cdot \frac{\sqrt{3}}{2}$

$A = \frac{\sqrt{3}}{4}(4-k^2)$

iii) Let slice thickness be δk

$\delta V = \frac{\sqrt{3}}{4}(4-k^2)\,\delta k$

$V = \int_{-2}^{2} \frac{\sqrt{3}}{4}(4-k^2)\,dk = 2\int_0^2 \frac{\sqrt{3}}{4}(4-k^2)\,dk$

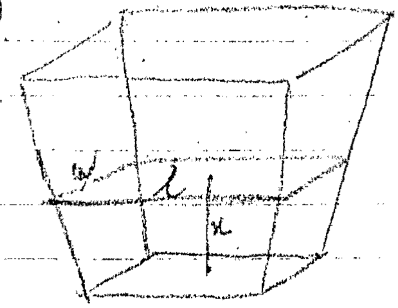
$= \frac{\sqrt{3}}{2}\left[4k - \frac{k^3}{3}\right]_0^2$

$= \frac{\sqrt{3}}{2}\left(8 - \frac{8}{3} - 0 + 0\right)$

$= \frac{\sqrt{3}}{2} \cdot \frac{16}{3}$

$V = \frac{8\sqrt{3}}{3}$ u^3

b)



length $= mx + b$

$x = 0 \quad l = 30$

$30 = b$

$x = 20 \quad l = 70$

$70 = 20m + 30$

$40 = 20m$

$2 = m$

$\therefore l = 2x + 30$

Width $= mx + b$

$x = 0 \quad W = 20$

$20 = b$

$x = 20 \quad W = 50$

$50 = 20m + 20$

$30 = 20m$

$\frac{3}{2} = m$

$W = \frac{3x}{2} + 20$

$$\text{Area} = (2x + 30)\left(\frac{3x}{2} + 20\right)$$

$$= 3x^2 + 40x + 45x + 600$$

$$= 3x^2 + 85x + 600$$

$$\delta V = (3x^2 + 85x + 600)\,\delta x$$

$$V = \int_0^{20} 3x^2 + 85x + 600 \; dx$$

$$= \left[x^3 + \frac{85x^2}{2} + 600x\right]_0^{20}$$

$$= 8000 + 17000 + 12000$$

$$V = 37000 \text{ cm}^3$$

$$= 37000 \text{ ml}$$

$$V = 37 \text{ litres}$$

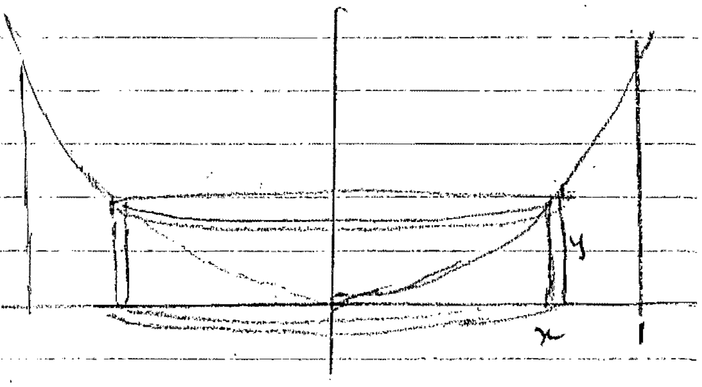
c) $V = \int 2\pi x y \; dx$

$$= 2\pi \int_0^1 x\left(e^{x^2} - 1\right) dx$$

$$= 2\pi \int_0^1 x e^{x^2} - x \; dx$$

$$= 2\pi \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2}\right]_0^1$$

$$= \pi\,(e - 1 - 1 + 0) = \pi(e - 2)\ u^3$$



Question 5

a) $P(x) = x^3 - 3x^2 + 9 = 0$

Put $w = \alpha^2 \quad \therefore \quad \alpha = \pm\sqrt{w}$

$\therefore \quad (\pm\sqrt{w})^3 - 3(\pm\sqrt{w})^2 + 9 = 0$

$\pm w\sqrt{w} - 3w + 9 = 0$

$\pm w\sqrt{w} = 3w - 9$

Square b/s $\quad w^3 = 9w^2 - 54w + 81$

$w^3 - 9w^2 + 54w - 81 = 0$

Similarly for β + γ

$\therefore$ Equation is $x^3 - 9x^2 + 54x + 81 = 0$

ii) $\therefore \quad \alpha^2 + \beta^2 + \gamma^2 = -\frac{b}{a} = 9$

α is a root of original eqn $\therefore \quad \alpha^3 - 3\alpha^2 + 9 = 0$

β .. .. .. $\quad \beta^3 - 3\beta^2 + 9 = 0$

γ .. .. .. $\quad \gamma^3 - 3\gamma^2 + 9 = 0$

Adding $\quad \alpha^3 + \beta^3 + \gamma^3 - 3(\alpha^2 + \beta^2 + \gamma^2) + 27 = 0$

$\alpha^3 + \beta^3 + \gamma^3 = 3(9) - 27 = 0$

b) $P(x)$ has a double root at $x = \alpha \quad \therefore \quad (x-\alpha)^2$ is a factor

Let $P(x) = (x-\alpha)^2 . Q(x)$

Differentiating using the product rule

$P'(x) = (x-\alpha)^2 . Q'(x) + Q(x) . 2(x-\alpha)'$

$= (x-\alpha)\{(x-\alpha)Q'(x) + 2Q(x)\}$

$\therefore$ $P'(x)$ has a factor of $x - \alpha$

$\therefore$ $P'(x)$ has a single zero at $x = \alpha$

c) $P(1) = 3$, & $P(2) = 5$

Dividing $P(x)$ by $(x-1)(x-2)$ the remainder is of degree 1,

Let it be $ax+b$

$$P(x) = (x-1)(x-2)\,Q(x) + (ax+b)$$

Sub $x=1$ $\quad 3 = a+b \quad$ — (1)

Sub $x=2$ $\quad 5 = 2a+b \quad$ — (2)

(2) − (1) $\quad 2 = a \quad \therefore \; b = 1$

$\therefore$ Remainder is $2x+1$

d) $P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$

For integer roots, root must divide $\frac{1}{1}$

Try $\quad P(1) = 1 - 2 + 3 - 4 + 1 \neq 0$

$\quad P(-1) = 1 + 2 + 3 + 4 + 1 \neq 0$

Neither $(x-1)$ nor $(x+1)$ is a factor

$\therefore$ no integer roots.

ii) $P(0) = 1$, $\quad P(1) = -2$

$P(x)$ is continuous, $P(0)$ & $P(1)$ are of opposite signs. $\therefore$ a root exists between 0 & 1.

iii) $\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$$= \left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = 2^2 - 2 \times 3$$

$$= 4 - 6 = -2$$

iv) Since $\Sigma\alpha^2$ is negative roots cannot all be real. Since one complex root exists there must be another — its conjugate as all coefficients are real. Therefore there are 2 complex roots and at least one real. As complex roots must be in conjugate pairs, 2 roots are complex & 2 are real.

Question 6

a) Consider $(x^2-1)^2 \geq 0$ (equal when $x=\pm 1$)

$\therefore \quad x^4 - 2x^2 + 1 \geq 0$

Add $3x^2$ to b/s

$x^4 + x^2 + 1 \geq 3x^2$

Provided $x^2 \neq 0$ divide b/s by x^2 since x^2 is always positive.

$\therefore \quad \dfrac{x^4+x^2+1}{x^2} \geq 3$ for all $x \neq 0$

$\lim_{x\to 0}\left(x^2 + 1 + \frac{1}{x^2}\right) \rightarrow \infty$ which > 3

b) $y = \dfrac{c^2}{x}$ $\quad \dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

at $x = ca$, $\dfrac{dy}{dx} = -\dfrac{1}{a^2}$

Gradient of tangent is $-\dfrac{1}{a^2}$ at A

Gradient of normal is a^2 at A

$\therefore$ Normal is $y - \dfrac{c}{a} = a^2(x - ca)$

$y = a^2x - ca^3 + \dfrac{c}{a}$

$y = a^2x + \dfrac{c}{a}(1-a^4)$

ii) For B solve normal with $y = \dfrac{c^2}{x}$

$\dfrac{c^2}{x} = a^2x + \dfrac{c}{a}(1-x^4)$

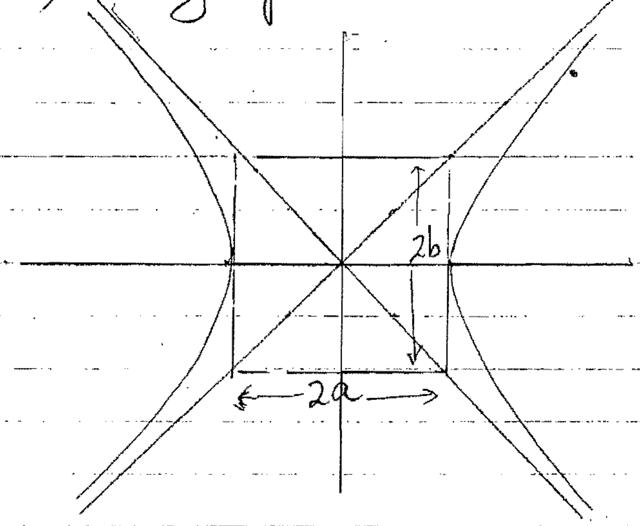
$\therefore \quad a^2x^2 + \dfrac{c}{a}(1-a^4)x - c^2 = 0$

Product of roots $= -\dfrac{c^2}{a^2}$ & roots are $x=ca$ & $x=cb$

$$\therefore c^2 ab = -\frac{c^2}{a^2}$$

$$b = -\frac{1}{a^3}$$

iii) Asymptotes become $y = \pm x$



Original Vertices $(\pm c, \pm c)$

Distance from O is $\sqrt{2}c$

$\therefore$ New vertices are $(\pm\sqrt{2}c, 0)$

$\therefore a = \sqrt{2}c$

Rectangular $\therefore b = \sqrt{2}c$

$\therefore$ Equation is $\frac{x^2}{2c^2} - \frac{y^2}{2c^2} = 1$

or $x^2 - y^2 = 2c^2$

c) Let $\angle DAP = \angle BAP = x$

" $\angle ABP = \angle PBC = y$

" $\angle BCR = \angle DCR = z$

" $\angle CDR = \angle RDA = w$

$2x + 2y + 2z + 2w = 360°$ (Angle Sum of quadrilateral)

$x + y + z + w = 180°$

$\therefore z + w = 180° - (x+y)$

$\angle BPS = x + y$ (Exterior $\angle$ of $\triangle APB$)

$\angle QPS = 180 - (x+y)$ (Straight $\angle$)

$= z + w$

But $\angle CRS = z + w$ (Exterior $\angle$ of $\triangle DRC$)

& $\angle QRS = 180 - (z+w)$ (Straight $\angle$)

$\therefore \angle QPS + \angle QRS = 180°$

But these are opposite $\angle$'s of quad PQRS

$\therefore$ PQRS is cyclic quadrilateral.

ii) Let $AD \parallel CB$

ii) $AD \parallel BC$ $\therefore 2x+2y=180°$ (cointerior $\angle$'s $AD \parallel BC$)

$\therefore x+y=90°$

$\therefore \angle BPS = \angle QPS = 90°$ (straight $\angle$)

$\therefore$ QS is a diameter (angle in a semicircle is a right angle.)

Question 7

a) i) Taking Downwards as positive & origin at height h

$\downarrow g$ $\uparrow R$

$R \propto v$

$R = -kv$

$$\frac{d^2x}{dt^2} = g - kv$$

ii)

$$\frac{dv}{dt} = g - kv$$

$$\frac{dt}{dv} = \frac{1}{g-kv} = \frac{-1}{k} \cdot \frac{-k}{g-kv}$$

$$t = -\frac{1}{k}\int \frac{-k}{g-kv}\,dv = -\frac{1}{k}\ln(g-kv) + c$$

$t=0,\ v=0$ $\quad \frac{1}{k}\ln g = c$

$$t = \frac{1}{k}\{\ln g - \ln(g-kv)\}$$

$$kt = \ln\frac{g}{g-kv}$$

$$\frac{g}{g-kv} = e^{kt} \quad \text{or} \quad \frac{g-kv}{g} = e^{-kt}$$

$$ge^{-kt} = g - kv$$

$$kv = g - ge^{-kt}$$

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

iii)

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = g - kv$$

$$\frac{dv}{dx} = \frac{g-kv}{v}$$

$$\frac{dx}{dv} = \frac{v}{g-kv} = -\frac{1}{k}\,\frac{g-kv-g}{g-kv}$$

$$\frac{dx}{dv} = -\frac{1}{k}\left\{1 - \frac{g}{g-kv}\right\}$$

$$\frac{dx}{dv} = -\frac{1}{k}\left\{1 + \frac{g}{k}\,\frac{-k}{g-kv}\right\}$$

$$x = -\frac{1}{k}\left\{v + \frac{g}{k}\ln(g-kv)\right) + c$$

$$x = -\frac{g}{k^2}\left\{\frac{kv}{g} + \ln(g-kv)\right\} + c$$

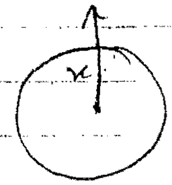
$x=0,\ v=0$ $\qquad C = -\frac{g}{k^2}\ln g$

$$x = -\frac{g}{k^2}\left\{\frac{kv}{g} + \ln(g-kv) - \ln g\right\}$$

$$x = \frac{g}{k^2}\left\{\ln g - \ln(g-kv) - \frac{kv}{g}\right\}$$

$$x = \frac{g}{k^2}\left\{\ln\left|\frac{g}{g-kv}\right| - \frac{kv}{g}\right\}$$

b) Taking upwards as positive & origin at centre of Earth.



$$\frac{d^2x}{dt^2} = -\frac{k}{x^2}$$

i) at $x = R$, $\frac{d^2x}{dt^2} = -g$

$$-g = -\frac{k}{R^2} \implies k = gR^2$$

ii)
$$\frac{d^2x}{dt^2} = \frac{-gR^2}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -gR^2x^{-2}$$

$$\frac{1}{2}v^2 = \frac{-gR^2x^{-1}}{-1} + c$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + c$$

$$v^2 = \frac{2gR^2}{x} + c_2$$

$x = R$, $v = u$ $\therefore$ $u^2 = \frac{2gR^2}{R} + c_2$

$$c_2 = u^2 - 2gR$$

$$\therefore v^2 = \frac{2gR^2}{x} + u^2 - 2gR$$

iii) Max ht when $v = 0$

$$\frac{2gR^2}{x} = 2gR - u^2$$

$$x = \frac{2gR^2}{2gR - u^2} = \text{greatest height}$$

iv) $x \to \infty$ $\therefore$ $2gR - u^2 = 0$

$$u^2 = 2gR$$

$u > \sqrt{2gR}$ for escape

$u > \sqrt{2 \times 9.8 \times 6367000}$ or $u > 11171$ m/s

or $u > 11.2$ Km/s.

Question 8

a) i) $I_n = \int \tan^n x \, dx$

$I_n = \int \tan^{n-2} x \tan^2 x \, dx$ $\qquad \tan^2 x = \sec^2 x - 1$

$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$

$= \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$

$I_n = \dfrac{1}{n-1} \tan^{n-1} x - I_{n-2}$

ii) $I_3 = \dfrac{1}{2} \tan^2 x - I_1$

$I_1 = \int \tan x \, dx = \int \dfrac{\sin x}{\cos x} \, dx$

$= - \ln \cos x$

$\therefore I_3 = \frac{1}{2} \tan^2 x + \ln \cos x$

$\therefore \int_0^{\frac{\pi}{4}} \tan^3 x \, dx = \left[\frac{1}{2} \tan^2 x + \ln \cos x \right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2} \tan^2 \frac{\pi}{4} + \ln \cos \frac{\pi}{4} - 0 - \ln \cos 0$

$= \frac{1}{2} + \ln \frac{1}{\sqrt{2}} - 0 - \ln 1$

$= \frac{1}{2} - \frac{1}{2} \ln 2$

b) i) $z^5 = 1 \quad (z \neq 1) \implies z^5 - 1 = 0$

$$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$

$z \neq 1 \quad \therefore z^4 + z^3 + z^2 + z + 1 = 0$

$\div z^2 \quad z^2 + z + 1 + z^{-1} + z^{-2} = 0 \quad \text{——— (1)}$

ii) $z^5 = 1 \quad = 1 \operatorname{cis}(0 + 2k\pi)$

$\therefore z = \sqrt[5]{1} \operatorname{cis} \frac{2k\pi}{5} \qquad k = 0, 1, 2, 3, 4$

But $z \neq 1 \quad \therefore z = \operatorname{cis} \frac{2k\pi}{5} \qquad k = 1, 2, 3, 4$

then $z + z^{-1} = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} + \cos\left(\frac{-2k\pi}{5}\right) + i \sin\left(\frac{-2k\pi}{5}\right)$

$= \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} + \cos \frac{2k\pi}{5} - i \sin \frac{2k\pi}{5}$

$= 2 \cos \frac{2k\pi}{5} \qquad k = 1, 2, 3, 4$

iii) If $x = z + z^{-1}$

$x^2 = z^2 + 2 + z^{-2} \implies z^2 + z^{-2} = x^2 - 2$

So (1) becomes $(z^2 + z^{-2}) + (z + z^{-1}) + 1 = 0$

$= x^2 - 2 + x + 1 = 0$

$x^2 + x - 1 = 0 \quad \text{——— (2)}$

iv) $k=1, \; x = z + z^{-1} = 2\cos\frac{2\pi}{5}$ | $k=2, \; x = 2\cos\frac{4\pi}{5} = -2\cos\frac{\pi}{5}$

$k=3, \; x = 2\cos\frac{6\pi}{5} = -2\cos\frac{\pi}{5}$ | $k=4, \; x = 2\cos\frac{8\pi}{5} = 2\cos\frac{\pi}{5}$

Hence solutions of (2) are $x = 2\cos\frac{2\pi}{5}$ & $-2\cos\frac{\pi}{5}$

Product of roots $= 2\cos\frac{2\pi}{5} \times -2\cos\frac{\pi}{5} = \frac{c}{a} = -1$

$-4\cos\frac{\pi}{5} \cdot \cos\frac{2\pi}{5} = -1$

$\cos\pi/5 \cdot \cos 2\pi/5 = \frac{1}{4}$